Paris School of Economics, Master 1 APE
Microeconomics 1, Problem Set 5

Michaelmas Term 2007-2008

Homework: finish the exercises of PS 4 (except exercise 1, question 2).

Exercise 1

Consider any consumer whose demand functions for the only two goods in the economy, 1 and 2, read:

$$x_1(p_1, p_2, w) = \alpha \frac{w}{p_1}, \quad (1)$$

$$x_2(p_1, p_2, w) = (1-\alpha)\frac{w}{p_2}. \quad (2)$$

Our aim is to find a utility function which generates these demand functions.

1. Check that these functions are homogenous, that they satisfy Walras' law and that the Slutsky matrix is negative semi-definite.
2. To recover a utility function U which generates $x_1(p_1, p_2, w)$ and $x_2(p_1, p_2, w)$, one can proceed as follows:
   (a) Use homogeneity to normalize w to 1 and express each price as a function of α, $x_1(p_1, p_2, w)$, $x_2(p_1, p_2, w)$.
   (b) Consider any indifference curve $U(x_1, x_2) = K$. For each given K, write $x_2 = \phi(x_1)$. Derive $d\phi(x_1)/dx_1$ and solve this differential equation.
   (c) What is the corresponding utility function?

Exercise 2

1. Let f and g be two production functions defined by:

$$f(x_1, x_2) = \min\{ax_1, bx_2\}, \text{ with } a, b > 0, \quad (3)$$

$$g(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \text{ with } 0 < \alpha < 1. \quad (4)$$

   Determine and then draw the isoquants in each of these cases.

2. Is it correct to state that "if a firm has constant returns to scale, then the marginal product of labour and capital depend only on the ratio of labour and capital"? Explain. Can you state the same for an homothetic production function? Explain.

3. Let the production function of a firm be given by $f(x_1, x_2) = \left(3\sqrt{x_1} + \sqrt{x_2}\right)^2$. The price of input 1 is 1 euro and the price of input 2 is 3 euros. What is the cheapest way to produce 16 units of output?

Exercise 3

Consider a Cobb-Douglas production function:

$$f(x) = x_1^{\alpha} x_2^{\beta}, \text{ with } \alpha > 0 \text{ and } \beta > 0. \tag{5}$$

Part I:

1. Sketch the production possibility set Z in each of the following cases:
   (a) $\alpha + \beta < 1$;
   (b) $\alpha + \beta > 1$;
   (c) $\alpha + \beta = 1$.
   Compute the marginal product, average product, and marginal rate of technical substitution, as function of x_1.
2. Is the production function homothethic and/or homogeneous?
3. What returns to scale are there?
4. Derive the factor demands $x_1(p, q)$ and $x_2(p, q)$.
5. Derive the supply function $y(p, w)$.
6. Find the marginal price effects. Confirm the signs (and where appropriate, relative magnitudes) of these effects.
7. Find the profit function.
8. Confirm that Hotelling's Lemma holds.

Part II:

1. Derive the conditional factor demand $h_1(w, q)$ and $h_2(w, q)$.
2. Find the cost function $c(w, y)$. Confirm is properties.
3. Sketch $c(w, y)$ as a function of y for each of the case $\alpha + \beta > 1$, $\alpha + \beta < 1$ and $\alpha + \beta = 1$. In addition, sketch marginal and average costs as a function of y for each of the three cases.
4. Why is it not necessary to assume that $\alpha + \beta < 1$ for cost minimization?
5. Confirm that Shepard's Lemma holds.

Exercise 4

Find the cost function for each of the following production functions:

1. $\min\left\{x_1, x_2^2\right\}$;
2. $2x_1 + 3x_2$;
3. $\max\left\{2x_1, 3x_2\right\}$;

Exercise 5

1. Assume that the cost function associated with some given technology is differentiable. Show that that the average cost is rising (falling) as marginal cost is higher (lower) than average cost.
2. Let $c(w,y) = (aw_1 + bw_2)y^{1/2}$ be a cost function. Derive its production function and draw a representative family of isoquants.
3. Let $f(x)$ be the production fonction of a firm with constant returns to scale technology. Assume each factor x_i $(i = 1, \ldots, I)$ is paid its *marginal product in value* $w_i = p\frac{\partial f(x)}{\partial x_i}$. Show that profits must be zero.