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Chapter 1 - Preference and choice

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Notations

Consider an individual (agent) facing a *choice set* X.

Definition (Choice set, "Consumption set")

X is a set of mutually exclusive choices.

Ex.1

$X = \{$ APE M1, another M1, other studies, stop studying $\}$

Ex.2 Non-divisible goods

$$X = \left\{ x = (x_1 \ldots x_M) \in \mathbb{R}_+^M, \ \sum_{k=1}^{M} x_k = 1, \ x_k \in \{0, 1\} \right\}$$

Ex.3 Divisible goods

$$X = \left\{ \mathbb{R}_+^M \right\}$$

Two approaches to model consumer behavior:

- A **preference-based approach** (dominant model)
  Assumes the decision maker has a preference relation over the set of possible choices that satisfies a rationality axiom.

- A **choice-based approach**
  Focuses directly on the choice behavior imposing consistency restrictions (parallels the rationality axiom of the preference-based approach).

The agent has a preference relation over X.
We impose rationality axioms on these preferences.
What consequences for the agent's choices?

Definition (Preference relation)

A preference relation (denoted $\succsim$) is a binary relation on X which compares couples $x, y \in X$.
$x \succsim y$ reads "x is ***preferred over or equivalent to*** *y".*
"x provides at least as much well-being as y".

$\Rightarrow$ Two other relations:

The *strict-preference* relation $\succ$
The *indifference* relation $\sim$

Definition (Strict-preference relation)

A strict-preference relation (denoted $\succ$) is defined as follows:

$$x \succ y \Leftrightarrow x \succsim y \text{ but } y \not\succsim x$$

*"x is (strictly) **preferred** over y".*
"x provides more well-being than y".

Definition (Indifference relation)

An indifference relation (denoted $\sim$) is defined as follows:

$$x \sim y \Leftrightarrow x \succsim y \text{ and } y \succsim x$$

*"x and y are **indifferent** ("the agent is"...)".*
"x and y provide the same well-being".

The theory (naturally) imposes a *rationality axiom*:

Definition (Rational preference (rationality axiom))

A preference relation $\succsim$ is ***rational*** *if it is:*

- *(i)* ***complete****:*

$$\forall x, y \in X,\ x \succsim y \text{ and } / \text{ or } y \succsim x$$

- *(ii)* ***transitive****:*

$$\forall x, y, z \in X,\ x \succsim y \text{ and } y \succsim z \Rightarrow x \succsim z$$

Discussion

No indecision nor changes in time. Precludes cyclical preferences (e.g. Condorcet paradox). "Framing" problem...

Proposition

If a preference relation $\succsim$ is rational (i.e. complete and transitive), then:

- $\succ$ *is*
  - *irreflexive* ($x \succ x$ *is impossible*)
  - *transitive* ($x \succ y$ *and* $y \succ z \Rightarrow x \succ z$)
- $\sim$ *is*
  - *reflexive* ($x \sim x$)
  - *transitive* ($x \sim y$ *and* $y \sim z \Rightarrow x \sim z$)
  - *symmetric* ($x \sim y \Rightarrow y \sim x$)
- $x \succ y \succsim z \Rightarrow x \succ z$

Other properties and graphic representations of preferences

Let's consider $X = \mathbb{R}_+^M$ (or $X = \mathbb{R}_+^2$ for graphs).

Monotonicity

Definition (Monotonicity)

A preference relation is monotone:

$$\Leftrightarrow \forall x, y \in X, \text{"} x \geq y \text{"} \Rightarrow x \succsim y$$

A preference relation is strongly monotone:

$$\Leftrightarrow \forall x, y \in X, \text{"} x \geq y \text{"} \textit{ and } x \neq y \Rightarrow x \succ y$$

Proposition

Strongly monotone ⇒ monotone.

Definition (Indifference curve, upper and lower contour sets)

- *An indifference curve is an equivalence class for the* $\sim$ *relation:* $I(\bar{x}) = \{x \in X : x \sim \bar{x}\}$
- *Upper contour set:* $P^{+}(\bar{x}) = \{x \in X : x \succsim \bar{x}\}$
- *Lower contour set:* $P^{-}(\bar{x}) = \{x \in X : \bar{x} \succsim x\}$

Example of monotone preferences (graph).

Local non-satiation

Definition (Local non-satiation)

$\succsim$ is locally non-satiated
$\Leftrightarrow \forall x \in X$, any neighborhood of x has a y such that $y \succ x$
$\Leftrightarrow \forall x \in X, \forall \varepsilon > 0, \exists y \in X$ such that $\|x - y\| \leq \varepsilon$ and $y \succ x$

Example of non-satiated preferences.
Example of satiated preferences ("thick" indifference curve).

Proposition

Strongly monotone $\Rightarrow$ locally non-satiated.

Convexity (a very important concept)

Definition (Convexity)

A preference relation is convex
$\Leftrightarrow$ *If* $x \succsim z$*, and* $y \succsim z$ *then* $\forall \lambda \in [0,1],\ \lambda x + (1-\lambda) y \succsim z$
$\Leftrightarrow$ *If* $x \succsim y$*, and* $y \succsim z$ *then* $\forall \lambda \in [0,1],\ \lambda x + (1-\lambda) y \succsim z$
$\Leftrightarrow$ $P^{+}(x)$ *is a convex set*
(if $y, z \in P^{+}(x)$ *then* $\forall \lambda \in [0,1],\ \lambda y + (1-\lambda) z \in P^{+}(x)$ *)*

Demonstration of equivalence: exercise.
Example of non-convex and **convex preferences** (graphs).

Definition (Strict convexity)

A preference relation $\succsim$ *is strictly convex*
$\Leftrightarrow \quad \forall x, y, z \in X,\ x \neq y,\ x \succsim z$ *and* $y \succsim z$
$\Rightarrow \forall \lambda \in]0,1[,\ \lambda x + (1-\lambda) y \succ z$

Example of strictly convex preferences.

Interpretation of convex preferences:

- love for diversity
- decreasing "marginal rate of substitution" (MRS). One needs an increasing quantity of a good in order to compensate for successive losses of another good.

Continuity (another important concept)

Definition (Continuity)

A preference relation $\succsim$ *is continuous*

$\Leftrightarrow\ \forall x \in X, P^{+}(\bar{x}) = \{x \in X : x \succsim \bar{x}\}$

and $P^{-}(\bar{x}) = \{x \in X : \bar{x} \succsim x\}$

are closed (they include their boundaries).

$\Leftrightarrow \succsim$ *is preserved under limits:*

(i.e. for any sequence of pairs $\{(x_n, y_n)\}_{n=1}^{\infty}$ *with* $x_n \succsim y_n\ \forall n,\ \lim x_n = x,$ *and* $\lim y_n = y$*, then* $x \succsim y$*)*

Important results derive from the continuity property.
Examples of non-continuous preferences ("jump" in the indifference curve).

Homotheticity

Definition (Homotheticity)

A preference relation is homothetic
$\Leftrightarrow$ *indifference curves are linked by a homothecy*
i.e. if $x \sim y$ *then* $\forall \alpha \geq 0, \ \alpha x \sim \alpha y$

Graphic illustration.

Question: Agents have preferences. How do they choose?
Choice criterion: choose an element of X that is "maximal" for $\succsim$.

Definition (Maximal element)

An element $x \in X$ is maximal for $\succsim$
$\Leftrightarrow \nexists y \in X$ *such that* $y \succ x$.

Existence? Unicity?

Proposition

If X is compact and $\succsim$ is continuous
$\Rightarrow$ there exists at least one maximal element.

Proposition

If X is convex and compact and $\succsim$ is strictly convex
$\Rightarrow$ $\succsim$ has a unique maximal element in X.

Utility functions are a fundamental concept.
A utility function provides a numerical value to each choice and allows for comparisons.

Definition (Utility function)

Consider a function $u : X \rightarrow \mathbb{R}$
u represents the preference relation $\succsim$ *if* $\forall\, x, y \in X$*:*
$x \succsim y \;\Leftrightarrow\; u(x) \geq u(y)$

Remarks:

- An infinity of utility functions can represent a given $\succsim$ (proof: composition with a *strictly-increasing* function).
- The properties of utility functions which are preserved for any strictly increasing transformation are called *ordinal*. *Cardinal* properties are those not preserved under such transformations.
- The preference relation associated with a utility function is an ordinal property. The numerical values associated with elements of X and the difference in utility between two elements of X are cardinal properties.

Proposition (Representation theorem)

If a rational preference relation (i.e. complete and transitive) $\succsim$ is continuous
$\Rightarrow$ *there is a continuous utility function $u(x)$ that represents $\succsim$.*

Remarks:

- The lexicographic preferences cannot be represented by a utility function.
- In the general case, utility functions need not be continuous.

Important properties of utility functions

Concavity

Definition (Concavity)

$u : X \to \mathbb{R}$ *is concave iff*

$$\forall x, y \in X,\ x \neq y,\ \forall \lambda \in]0,1[,$$
$$u(\lambda\, x + (1-\lambda)\, y) \geq \lambda\, u(x) + (1-\lambda)\, u(y)$$

Definition (Strict concavity)

$u : X \to \mathbb{R}$ *is strictly concave iff*

$$\forall x, y \in X,\ x \neq y,\ \forall \lambda \in]0,1[,$$
$$u(\lambda\, x + (1-\lambda)\, y) > \lambda\, u(x) + (1-\lambda)\, u(y)$$

- Graphic examples with $X = \mathbb{R}$ (arcs below the graph).
- Strict concavity $\Rightarrow$ concavity.
- Concavity is a *cardinal* property.
- Other mathematical definitions to characterize concave functions (see *Exercise class*):

Proposition

- *A twice-differentiable function* $u : X \rightarrow \mathbb{R}$ *is concave*
  $\Leftrightarrow$ *the Hessian matrix is negative semi-definite.*
- *A twice-differentiable function* $u : X \rightarrow \mathbb{R}$ *is strictly concave*
  $\Leftarrow$ *the Hessian matrix is negative definite.*

Quasi-concavity

Definition (Quasi-concavity)

$u \; : \; X \to \; \mathbb{R}$ *is quasi-concave*

$\Leftrightarrow$

$$\forall x, y \in X, \; x \neq y, \; \forall \lambda \in]0,1[,$$
$$u(\lambda\, x + (1-\lambda)\, y) \geq \; \min\{u(x), u(y)\}$$

$\Leftrightarrow$

$\{y \in X : u(y) \geq u(x)\}$ *is convex* $\forall x$.

Remark: // with definition of convex preferences ($P^{+}(x)$ *convex*).

Definition (Strict quasi-concavity)

$u \;:\; X \rightarrow \mathbb{R}$ *is strictly quasi-concave*

$\Leftrightarrow$

$$\forall x, y \in X,\ x \neq y,\ \forall \lambda \in]0,1[,$$
$$u(\lambda\, x + (1-\lambda)\, y) > \min\{u(x), u(y)\}$$

- Graphic examples (indifference curves).
- Quasi-concavity is an ordinal property ($\neq$ concavity which is cardinal).

Quasi-concavity is "weaker" than concavity:

Proposition

We have the following implications:

Concave $\Rightarrow$ quasi-concave
Strictly concave $\Rightarrow$ strictly quasi-concave
Strictly quasi-concave $\Rightarrow$ quasi-concave

But we do not need more!

Proposition

Consider $\succsim$ represented by u

- $\succsim$ *is **convex*** $\Leftrightarrow$ u *is **quasi-concave***
- $\succsim$ *is strictly convex* $\Leftrightarrow$ u *is strictly quasi-concave*

Another key concept: the **Marginal Rate of Substitution**

Definition (Marginal Rate of Substitution (MRS))

Consider two goods, 1 *and* 2*, and a consumption bundle* $\overline{x} = (\overline{x}_1, \overline{x}_2)$*. The "marginal rate of substitution of good* 1 *for good* 2*", denoted* MRS_{12} *tells the amount of good* 2 *that the consumer must be given to compensate for a one-unit marginal reduction in his consumption of good* 1 *(i.e. in order to keep the same utility* $u(\overline{x})$ *or, equivalently, to remain on the same indifference curve* $I(\overline{x})$*):*

$$MRS_{12}(\overline{x}) = -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}(\overline{x}) = \frac{dx_2}{dx_1}(\overline{x})$$

Remarks

- In the general case, for any k and l goods, the MRS writes:

$$MRS_{kl}(\overline{x}) = -\frac{\frac{\partial u}{\partial x_k}}{\frac{\partial u}{\partial x_l}}(\overline{x}) = \frac{dx_l}{dx_k}(\overline{x})$$

- It does not depend on the chosen utility function.
- It is a local concept which varies along the indifference curve.
- It is a subjective exchange rate.
- If, in a given situation, two individuals have different MRS, they will find it advantageous to trade products.

To sum up

Proposition

The following assertions are equivalent:

Preferences are convex
$\Leftrightarrow$ *The utility function is quasi-concave*
$\Leftrightarrow$ MRS_{12} *decreases (in absolute value) with* x_1

Coming back to the choice criterion

- Under some conditions, $\succsim$ is represented by u
- Under some conditions, $\exists$! maximal element of X for $\succsim$
- Choosing $x \in X$ maximal for $\succsim$ $\Leftrightarrow$ choosing $x \in X$ which maximizes u (see Chapter 3: the classical demand theory) $\Rightarrow$ rational agents are "maximizers".

- The choice-based approach to decision theory has an empirical flavor.
- The focus is directly on choices, trying to impose a consistent structure: the **Weak Axiom of Revealed Preferences** (WA).
- The WA parallels the rationality axiom in the preference-based approach)

Definition (Choice structure)

Consider:
A choice set X (e.g. $X = \{a, b, c\}$)
A family $\mathcal{F}$ of non-empty subsets of X represents the situations an agent might be facing.
(e.g. $\mathcal{F} = \{\{a, b\}, \{a, b, c\}\}$

- *A choice rule, defined on $\mathcal{F}$, says which elements are chosen or can be chosen by the agent.*
  (e.g. $C(\{a, b\}) = \{a\}$ or $C(\{a, b, c\}) = \{a, b\}$
- *$(\mathcal{F}, C)$ forms a choice structure.*

Question: What reasonable restrictions shall we impose for choices to be consistent?
Answer: the Weak Axiom of Revealed Preferences.

Intuition of the WA: if a is chosen while b is available, then this choice reveals a preference for a over b. Thus, there must not exist another situation in which b is chosen while a is available but a is not also chosen.

Definition (WA)

A choice structure $(\mathcal{F}, C)$ satisfies the WA iff

$$\forall B \in \mathcal{F} : B \supseteq \{a, b\} \text{ and } a \in C(B) \;\Rightarrow$$

$$\{\forall B' \in \mathcal{F} \quad B' \supseteq \{a; b\} \text{ and } b \in C(B') \Rightarrow a \in C(B')\}$$

Does the previous example satisfy the WA?

The WA can be restated in terms of preference relations:

Definition (The revealed preference relation)

The revealed preference relation $\succsim^*$ is defined on X by:

$$a \succsim^* b \Leftrightarrow \exists B \in \mathcal{F} \ : \ a, b \in B \text{ and } a \in C(B)$$

It reads "a is revealed at least as good as b"

Definition (The revealed dominance relation)

The revealed dominance relation $\succsim^{**}$ is defined on X by

$$a \succsim^{**} b \Leftrightarrow \exists B \in \mathcal{F} \ : \ a, b \in B, \ a \in C(B) \text{ and } b \notin C(B)$$

It reads "a is revealed preferred over b"

Definition (WA, second definition)

A choice structure $(\mathcal{F}, C)$ satisfies the WA iff

$$a \succsim^{*} b \Rightarrow \neg\, [b \succsim^{**} a]$$

If a is revealed at least as good as b then b cannot be revealed preferred over a.

Example.

How do the two approaches relate?

From preferences → choices

Question: Consider an agent with a rational preference relation $\succsim$ confronted to $\mathcal{F}$. Can we derive from $\succsim$ a choice structure that *satisfies the WA*?

Answer: Yes. It suffices to consider the choice rule $C^*(B, \succsim) = \{x \in B : \forall y \in B,\ x \succsim y\}$ which comes down to always choose the maximal element for $\succsim$. It always satisfies the WA.

Proof.

How do the two approaches relate?

From choices → preferences

Question: Consider a choice structure $(\mathcal{F}, C)$ that satisfies the WA. Can we define a *rational preference relation* $\succsim$ that generates this choice structure (i.e. such that the selection of maximal elements for $\succsim$ will coincide with the choice structure $(\mathcal{F}, C)$)?

Answer: In some particular cases only.

Example: a choice structure satisfying the WA but that cannot be rationalized.

How do the two approaches relate?

- A rational preference relation always "generates" a choice structure which satisfies the WA.
- However, it may not be possible to "rationalize" a choice structure which satisfies the WA.
- The standard consumer theory focuses on the preference-based approach.