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Chapter 3 - The classical demand theory (preference-based approach)

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Introduction

Proposition

- ***Representation theorem***:
  If $\succsim$ is rational and continuous, it can be represented by a (continuous) utility function u.

- ***Choice criterion***:
  Choosing the maximal element for $\succsim$ $\Leftrightarrow$ Maximizing u

Notations

- Consumption set: $X \subset \mathbb{R}_+^M$
- Consumption bundle: $x \in X$
- (Exogenous) price vector: $p = (p_1, \dots p_M) \in \mathbb{R}_+^M$
- (Exogenous) income or wealth: $w \in \mathbb{R}_+$

The utility maximization program

The Marshallian (Walrasian) demand function

Definition (Marshallian (Walrasian) demand function)

The Marshallian demand function is the solution denoted ***x(p,w)*** *of the following maximization program:*

$$\mathcal{P} \left\{ \begin{array}{c} \underset{x}{Max}\ u(x) \\ s.t.\ p \cdot x \leq w \end{array} \right.$$

Remark:

$$p \cdot x \leq w \;\Leftrightarrow\; x \in\; B(p,w) = \left\{x \in \mathbb{R}_+^M \colon p \cdot x \leq w\right\})$$

$$x(p,w) = \underset{x \in B(p,w)}{Argmax}\ u(x)$$

Properties of the Marshallian demand function

- (i) If u is continuous, there exists a solution to the program (since $B(p, w)$ is compact).
- (ii) If u is strictly quasi-concave, then $x(p, w)$ is unique.
- (iii) $x(p, w)$ is homogeneous of degree 0.
- (iv) If u represents locally non-satiated preferences, $x(p, w)$ satisfies Walras'law.

Example:

$$\begin{cases} \underset{x_1, x_2}{Max} \ u(x_1, x_2) = (x_1 + 4)(x_1 + x_2) \\ \quad s.t. \ 3x_1 + 2x_2 \leq 10 \end{cases}$$

- A quicker way to solve a consumer's program. If x^* is the solution of the consumer's program, then:

$$\left|MRS_{i,j}(x^*)\right| = \frac{p_i}{p_j} \;\; \forall \; i,j$$

Proof: optimality condition for an interior solution using the Lagrangian.

- **Interpretations**:
(i) An additional Euro can be spent on any good.
(ii) Equality between the subjective and objective exchange rates.

- **Remark**: The Lagrange multiplier is the marginal utility of wealth or income (mathematical property of the Lagrange multiplier).

- Back to the **Example**
- **Graphic illustration**

Interior solution (MRS $=$ price ratio)

Corner solution (MRS $\neq$ price ratio)

$x(p, w)$ single valued (u strictly quasi-concave)

The indirect utility function

Definition (Indirect utility function)

Consider $w > 0$ and $p_k > 0 \ \forall k = 1 \dots M$.

The indirect utility function ***v(p,w)*** *is the utility obtained when consuming the Marshallian demand:*

$$v(p, w) = u(x(p, w))$$

Remark:

$$u(x(p, w)) = \underset{x \in B(p,w)}{Max} \ u(x)$$

Properties of the indirect utility function

If u is continuous over X and represents locally non-satiated $\succsim$, then $v(p, w)$ is:

- (i) Homogeneous of degree 0
- (ii) Strictly increasing in w
- (iii) Decreasing in p
- (iv) Quasi-convex
  (i.e. $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex $\forall \bar{v} \in \mathbb{R}$).

Proof.

Remark: Quasi-convexity.
Illustration of $v(p, w)$'s quasi-convexity.

The dual program: the expenditure minimization program

The Hicksian demand function

Definition (Hicksian demand function)

The Hicksian demand function (or "compensated demand function") is the solution ***h(p,u)*** *of the following optimization program:*

$$\mathcal{D} \left\{ \begin{array}{c} \underset{x}{Min}\, p \cdot x \\ s.t.\ u(x) \geq u \end{array} \right.$$

Remark: $h(p,u) = \underset{x \in \{x:\ u(x) \geq u\}}{Argmin} \quad p \cdot x$ **Graph.**

Properties of the Hicksian demand function

$h(p, u)$ satisfies the following properties:

- (i) Homogeneity of degree 0 in p.
- (ii) $u(h(p, u)) = u$ if $h(p, u)$ is unique
  (otherwise, we can write $u(h) = u \ \ \forall h \in h(p, u)$).
- (iii) If u represents convex preferences,
  then $h(p, u)$ is a convex part of $\mathbb{R}^M_+$.

  If u represents strictly convex preferences,
  then $h(p, u)$ is unique.

Proof.

The expenditure function

Definition (Expenditure function)

Consider $u > 0$ *and* $p_k > 0 \ \forall k = 1 \ldots M$.

The expenditure function ***e(p,u)*** *is the total expenditure when consuming* $h(p, u)$*:*

$$e(p, u) = p \cdot h(p, u)$$

Remark:

$$e(p, u) = \underset{x \in X:\ u(x) \geq u}{Min} \ p \cdot x$$

Properties of the expenditure function

$e(p, u)$ satisfies the following properties:

- (i) Homogeneous of degree 1 in p
- (ii) Strictly increasing in u
- (iii) Increasing in $p_k \ \ \forall k = 1 \ldots M$
- (iv) Concave in p
- (v) Continuous in p and u

Proof.

Illustration of $e(p, u)$'s concavity in p and interpretation.

The link between the two approaches

Proposition (Equivalence of "primal" and dual programs)

- *Consider* x^* *solution of* $\begin{cases} \underset{x}{Max}\ u(x) \\ s.t.\ p \cdot x \leq w \end{cases}$

  then x^* *is solution of* $\begin{cases} \underset{x}{Min}\ p \cdot x \\ s.t.\ u(x) \geq u(x^*) \end{cases}$

- *If* $x^* = x(p, w)$ *then* $h(p, u(x^*)) = x^*$

Graph.

Proposition (Equivalence of "primal" and dual programs, ctn'd)

- *Consider* h^* *solution of* $\left\{ \begin{array}{c} \underset{x}{Min}\ p \cdot x \\ s.t.\ u(x) \geq u \end{array} \right.$

  then h^* *is solution of* $\left\{ \begin{array}{c} \underset{x}{Max}\ u(x) \\ s.t.\ p \cdot x \leq p \cdot h^* \end{array} \right.$

- *If* $h^* = h(p, u)$ *then* $x(p, p \cdot h^*) = h^*$

Graph.

Implications

Proposition (Duality relations)

- $x(p, w) = h(p, v(p, w))$
- $x(p, e(p, u)) = h(p, u)$
- $v(p, e(p, u)) = u$
- $w = e(p, v(p, w))$

Remark

$v(p, e(p, u)) = u$ and $w = e(p, v(p, w))$ mean that the expenditure function and the indirect utility function describe two sides of the same choice problem. One can be obtained by *inverting* the other.

Hicksian compensation

We have:

$$h(p,u) = x(p, \underbrace{e(p,u)}_{w})$$

When prices vary, $h(p,u)$ indicates how the Marshallian demand would adjust if wealth was modified to ensure that the consumer still obtains utility u (i.e. adjusting the consumer's wealth so that the new wealth exactly enables him to buy a quantity that will yield the utility level u when spent efficiently).

Definition (Hicksian compensation)

Hicksian compensation is the variation in wealth Δw following a variation in price ($p \rightarrow p'$) such that the utility-maximizing consumer keeps the same initial utility $v(p, w)$.

Graph.

Hicksian compensation $\neq$ Slutsky compensation.

Proposition (The law of compensated demand)

If u is strictly quasi-concave and $p \gg 0$, then:

$$\forall p, p' \in \mathbb{R}_{+}^{M}, (p' - p) \cdot (h(p', u) - h(p, u)) \leq 0$$

Implication for good l : $\Delta p_l \Delta h_l \leq 0$

Compensated demand is a decreasing function of price.

This is not necessarily true for the Marshallian demand (the wealth effect might dominate the price effect as for Giffen goods).

Important relations between the Marshallian demand, the Hicksian demand, the indirect utility function and the expenditure function

Assume strictly convex $\succsim$, continuous, locally non-satiated

The compensated demand and the expenditure function

Lemma (Shephard's lemma)

$$\forall k \in \{1, \ldots, M\} \;\; h_k(p, u) = \frac{\partial e(p, u)}{\partial p_k}$$

A useful way of calculating $h(p, u)$.

Proof.

The Marshallian demand and The Hicksian demand

Proposition (The Slutsky Equation)

Let us focus on how the price variation of a good affects the demand for another good:

$\forall p, w$ $\quad$ (*denoting* $u = v(p,w)$) $\quad$ *and* $\quad \forall k, j \in \{1, \dots, M\}$:

$$\frac{\partial x_k}{\partial p_j}(p,w) = \underbrace{\frac{\partial h_k}{\partial p_j}(p,u)}_{substitution\ effect} - \underbrace{\frac{\partial x_k}{\partial w}(p,w) \cdot x_j(p,w)}_{wealth\ effect}$$

Proof. Graph.

Interpretation: decomposition of substitution and price effects

$$\Delta x_k \simeq \frac{\partial h_k}{\partial p_j}(p,u)\cdot\Delta p_j \; - \; \frac{\partial x_k}{\partial w}(p,w)\cdot x_j(p,w)\cdot\Delta p_j$$

A $\rightarrow$ C. The **substitution effect** measures the compensated response to the price variation (i.e. facing the new price ratio but remaining on the same indifference curve).

C $\rightarrow$ B. The **wealth effect** measures the response to the change in *real* wealth following the price variation (i.e. facing the shift in the budget constraint).

Implications

A good k is normal $\Leftrightarrow \frac{\partial h_k}{\partial p_j} - \frac{\partial x_k}{\partial p_j} \geq 0$

It is inferior $\Leftrightarrow \frac{\partial h_k}{\partial p_j} - \frac{\partial x_k}{\partial p_j} < 0$

Proof. Graph. Interpretation.

Hence, only inferior goods can be Giffen goods.

Proof.

Definition (The Slutsky substitution matrix)

The subsitution matrix or Slutsky matrix is the $M \times M$ matrix of the partial second derivatives of the expenditure function with respect to prices (i.e. the derivatives of the Hicksian demand, measuring substitution effects)

$$S(p, w) = \left(s_{k,j}(p, w)\right)_{k,j \,\in\, \{1,\ldots,M\}}$$

with

$$s_{k,j}(p, w) = \frac{\partial^2 e(p, u)}{\partial p_k p_j} = \frac{\partial h_k(p, u)}{\partial p_j}$$

$$= \frac{\partial x_k}{\partial p_j}(p, u) + \frac{\partial x_k(p, w)}{\partial w} \cdot x_j(p, w)$$

Remarks

- $s_{k,j}(p, w)$ can be calculated from observations (i.e. the consumptions $x(p, w)$ that are observed given prices and wealth).

- Using Shepard's lemma and the concavity of the expenditure function, it can be shown that when demand is generated by preference maximization, the Slutsky substitution matrix is negative semidefinite and symmetric.

- In real situations, one may want to check the symmetry of the Slutsky matrix. If not, then it is impossible to find preferences that rationalize demand.

Marshallian demand and indirect utility

Proposition (Roy's Identity)

Assume $v(p, w)$ differentiable and $p \gg 0$. Then, $\forall k = 1, \ldots, M$, we have:

$$x_k(p, w) = -\frac{\frac{\partial v}{\partial p_k}(p, w)}{\frac{dv}{dw}(p, w)}$$

Proof. Parallel with Shepard's lemma. A quick and indirect way to calculate the Marshallian demand.

Summary of Duality

Primal program $\mathcal{P}$

Marshallian demand: $x(p, u)$
Indirect utility: $v(p, w)$
Roy's identity: $x_k(p, w) = -\frac{\frac{\partial v}{\partial p_k}(p,w)}{\frac{dv}{dw}(p,w)}$

Dual program $\mathcal{D}$

Hicksian demand: $h(p, u)$
Expenditure function: $e(p, u)$
Shepard's lemma: $h_k(p, u) = \frac{\partial e(p,u)}{\partial p_k}$

Duality relations

$$x(p,w) = h(p, v(p,w))$$
$$x(p, e(p,u)) = h(p,u)$$
$$v(p, e(p,u)) = u$$
$$w = e(p, v(p,w))$$

Slutsky equation: $\frac{\partial x_k}{\partial p_j}(p,w) = \frac{\partial h_k}{\partial p_j}(p,u) - \frac{\partial x_k}{\partial w}(p,w) \cdot x_j(p,u)$

Overall diagram.

The integrability problem

The utility maximization program (i.e. the consumer's rational behavior in the preference-based approach) leads to:

(i) $x(p, w)$ homogeneous of degree 0
(ii) $x(p, w)$ satisfies Walras' law
(iii) $x(p, w)$ is such that the Slutsky substitution matrix

$$S(p, w) = \left(\frac{\partial x_k}{\partial p_j}(p, u) + \frac{\partial x_k(p, w)}{\partial w} \cdot x_j(p, w) \right)_{kj}$$

is negative semidefinite and symmetric.

- **Question**:

  Reciprocally, if we observe $x(p, w)$ satisfying (i), (ii) and (iii), can we find $\succsim$ that rationalize $x(.,.)$?

- **Answer**:

  Yes. Use $h_k = \frac{\partial e}{\partial p_k}$ and integrate (cf. exercise class).

Money metric measures of welfare changes

- We want to evaluate the change in well-being following a price variation.
- This well-being variation can be measured in monetary units.
- Assume an initial price vector $p^0 \rightarrow p^1$ while wealth w remains constant. The consumer's well-being increases iff:

$$\Delta = v(p^1, w) - v(p^0, w) \geq 0$$

- Consider $\bar{p} \gg 0$ a given price vector. We want to construct a money metric indirect utility function using $\bar{p}$.
- $e(\bar{p}, v(p, w))$ measures the minimum expenditure needed (i.e. the minimum wealth) necessary to obtain utility $v(p, w)$ when prices prices are given by $\bar{p}$.
- It is a strictly increasing transformation of $v(p, w)$ (since the function $e(p, u)$ is strictly increasing in u). It is thus another utility function that represents $\succsim$ but is expressed in monetary units.

Proposition (Money metric measure of a welfare change)

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$$

is a monetary measure of the welfare change associated with the price variation ($p^0 \rightarrow p^1$).

Graph.

Problem:

What simple choice for $\bar{p}$?

There are two natural candidates: p^0 or p^1.

The Equivalent Variation: choice of $\bar{p} = p^0$

Definition (Equivalent Variation)

The Equivalent Variation (EV) is:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0)$$

$$= e(p^0, u^1) - w$$

Notations: $u^0 = v(p^0, w)$ *and* $u^1 = v(p^1, w)$

Consider a price variation beneficial to the consumer. There are several ways to view the EV:

- The *EV* is the minimum (resp. maximum) amount of money that the consumer would accept to receive (resp. to pay) to avoid (resp. to incur) the price variation.
- The consumer is indifferent between a variation in wealth equal to *EV* and the price variation.
- *EV* measures the net wealth variation which offers the consumer the same final level of utility u^1 for the initial price level p^0.
- Hence, an equivalent definition is:

$$v(p^0, w + EV) = u^1$$

Proof. **Graph.**

The Compensating Variation: choice of $\bar{p} = p^1$

Definition (Compensating Variation)

The Compensating Variation (CV) is:

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0)$$

$$= w - e(p^1, u^0)$$

There are several ways to view the CV:

- It is the maximum (resp. minimum) amount of money that the consumer would accept to pay (resp. to receive) to incur (resp. to avoid) the price variation.

- The *CV* measures the net wealth variation that makes the consumer keep the same initial utility level u^0 in spite of the price variation.

- Hence, an equivalent definition is:

$$v(p^1, w - CV) = u^0$$

Proof. **Graph.**

Relations between EV, CV and the Hicksian demand

Assume that only the price of good 1 varies from p_1^0 to p_1^1.
Using $e(p^0, u^0) = w = e(p^1, u^1)$ and Shepard's lemma, we get:

$$EV = \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, \ldots, P_M^0, u^1)\; dp_1$$

$$CV = \int_{p_1^1}^{p_0^1} h_1(p_1, p_2^0, \ldots, P_M^0, u^0)\; dp_1$$

Proof.

Implications and graphic illustration:

Normal good $\Rightarrow EV > CV$

Inferior good $\Rightarrow CV > EV$

Implications and graphic illustration (continued):

Proposition (The consumer's Marshallian surplus)

- *The change in Marshallian consumer surplus is:*

$$\Delta \mathcal{S} = \int_{p_1^1}^{p_1^0} x_1(p_1, p_2^0, \ldots, p_M^0, w)\, dp_1$$

- *$\Delta \mathcal{S}$ is comprised between CV and EV and can be used to measure welfare changes (It can be computed from observable data).*

Implications and graphic illustration (continued):

Definition (The consumer's Marshallian surplus)

Definition of Marshallian consumer surplus (graph).

Discussion

Quasilinear utility functions

Aggregating individual demands

Proposition (Aggregating demands)

If $x_i^d(p)$ is agent i's demand, then $\sum_{i=1}^{N} x_i^d(p)$ is the aggregate demand