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Chapter 4 - Production

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The production activity

The production set

Let us consider M goods (potentially *inputs* or *outputs* or both, i.e. *"netputs"*).

Definition (Net output)

For good j:

$$y_j = \underbrace{y_j^o}_{output} - \underbrace{y_j^i}_{input}$$

Definition (Production plan)

A *production plan* *is a vector* $y = (y_j)_{1 \leq j \leq M}$ *of net outputs.*

- $y_j > 0$: *good* j *is in net production.*
- $y_j < 0$: *good* j *is in net input.*

Example: $M = 5, \; y = (-5, 3, -6, 3, 0)$

Definition (Production set)

The ***production set*** *Y is the set of all possible production plans.*

$$Y \subset \mathbb{R}^M$$

Example: $M = 2$

Definition (Efficient production plan)

A production plan $y \in Y$ is ***efficient*** *iff:*

$$(y + \mathbb{R}^M_+) \cap Y = \{y\}$$

Interpretation: The quantities that are produced cannot be increased without using more inputs (inputs are efficiently used).

Definition (Efficiency boundary)

The *efficiency boundary* $E(Y)$ *is the set of efficient production plans:*

$$E(Y) = \left\{ y \in Y : (y + \mathbb{R}_+^M) \cap Y = \{y\} \right\}$$

Graph.

Specialized firms

Definition (Specialized firm)

A firm is specialized if the set of goods can be partitioned into a set of outputs and a set of inputs.

Graphs.

Definition (Production correspondence)

*Consider a specialized firm producing p outputs from L inputs (with $p + L = M$) and let $\mathbf{q} = (q_1, \ldots, q_p)$ denote an **output vector** and $\mathbf{x} = (x_1, \ldots, x_L)$ an **input vector**.*

A production correspondence is a multi-valued application:

$$\varphi\colon \mathbb{R}_+^L \to \mathbb{R}_+^p$$

such that

$$\underbrace{x = (x_1, \ldots, x_L)}_{\text{input vector}} \mapsto \varphi(x) = \{q \in \mathbb{R}_+^p : (-x, q) \in E(Y)\}$$

The case of a single-product firm

Definition (Production function)

In the case of a single-product (specialized) firm ($p = 1$), a *production function* *is an application:*

$$f \colon \mathbb{R}_+^L \to \mathbb{R}_+$$

such that

$$\underbrace{x = (x_1, \ldots, x_L)}_{input\ \ vector} \mapsto f(x) = \max_q \{q \colon (-x, q) \in Y\}$$

Remark: $(-x, f(x)) \in E(Y)$.

Non-specialized firms

Definition (Transformation function)

In the case of a non-specialized firm, the production set Y can be described using a transformation function*, i.e. a function:*

$$T : \mathbb{R}^M \rightarrow \mathbb{R}_-$$

$$\underbrace{y = (y_1, \ldots, y_M)}_{net\ \ output\ \ vector} \mapsto T(y) = \begin{cases} 0 \text{ if } y \ \in \ E(Y) \\ < 0 \text{ otherwise} \end{cases}$$

Graph.

Definition (Marginal Rate of Transformation)

If $T(.)$ is differentiable and if $T(\overline{y}) = 0$, the Marginal Rate of Transformation of good l for good k is:

$$MRT_{l,k}(\overline{y}) = -\frac{\frac{\partial T(\overline{y})}{\partial y_l}}{\frac{\partial T(\overline{y})}{\partial y_k}}$$

$MRT_{l,k}(\overline{y})$ indicates how much the (net) output of good k can increase if the firm decreases the (net) output of good l by one marginal unit (while remaining on the efficiency boundary, i.e. for a given technology).

Graph.

Factor and product isoquants

Inputs (factors)

Here, we consider a specialized firm and a given production objective.

Definition (Set of required inputs)

*Consider a specialized firm producing p outputs with L inputs. Denote $\mathbf{q} = (q_1, \ldots, q_p) \in \mathbb{R}_+^p$ a given **output vector**.*

The set of required inputs to produce q is:

$$V(q) = \left\{ \underbrace{x = (x_1, \ldots, x_L)}_{input\ \ vector} \in \mathbb{R}_+^L : (-x, q) \in Y \right\}$$

Definition (Factor isoquant)

The factor isoquant (for a given production objective $q \in \mathbb{R}^p_+$) is the set of input vectors that can be efficiently used to produce q. It is given by the following correspondence:

$$I \colon \mathbb{R}^p_+ \to \mathbb{R}^L_+$$

$$q = (q_1, \ldots, q_p) \mapsto I(q) = \left\{x \in \mathbb{R}^L_+ : (-x, q) \in E(Y)\right\}$$

Remark: If $\# I(q) > 1$ then inputs can be substituted in the production of q. Otherwise, inputs are complements.

The case of a single-product firm

Definition (Factor isoquant for a single-output firm)

In the case of a single-output firm, the definition is:

$$I \colon \mathbb{R}_+ \rightarrow \mathbb{R}_+^L$$

$$q \mapsto I(q) = \left\{ x \in \mathbb{R}_+^L \colon x \in V(q) \text{ and } x \notin V(q') \; \forall \, q' > q \right\}$$

Remark: $I(q)$ makes it possible to produce q but not more.
Graphs.

Definition (Marginal Rate of Technical Substitution)

Consider a single-output firm with a production function $q = f(x_1, \ldots, x_L)$.

The Marginal Rate of Technical Substitution of input l for input k indicates the additional amount of input k that must be used to keep output constant when the amount of input l is decreased marginally.

$$MRTS_{l,k}(\overline{x}) = -\frac{\frac{\partial f(\overline{x})}{\partial x_l}}{\frac{\partial f(\overline{x})}{\partial x_k}}$$

Graph with $L = 2$ (slope).

- $MRTS_{l,k}$ is the renaming of $MRT_{l,k}$ in the special case of a single-output technology.
- $\frac{\partial f(\overline{x})}{\partial x_l}$ is the marginal productivity of input l at $\overline{x}$. A frequent assumption is "decreasing marginal productivity":

$$\frac{\partial f}{\partial x_l} > 0 \text{ and } \frac{\partial^2 f}{\partial {x_l}^2} < 0$$

- If two firms have different MRTS, then inputs can be reallocated so that a firm will produce more while the other will not produce less (cf. Pareto-optimality in Chapter 5).
- Cobb-Douglas example:

$$f(x_1, x_2) = x_1^{\alpha} x_2^{\beta} \text{ with } \alpha > 0 \text{ and } \beta > 0$$

Outputs (products)

Here, we consider a given input utilization objective.

Definition (Set of feasible outputs)

Consider a specialized firm producing p outputs with L inputs. Denote $\boldsymbol{x} = (x_1, \ldots, x_L) \in \mathbb{R}_+^L$ *a given* ***input vector****.*

The *set of feasible outputs* *using x is:*

$$H(x) = \left\{ \underbrace{q = (q_1, \ldots, q_p)}_{output\ \ vector} \in \mathbb{R}_+^p : (-x, q) \in Y \right\}$$

Definition (Product isoquant)

The product isoquant *(for a given input utilization objective $x \in \mathbb{R}_+^L$) is the set of output vectors that can be* efficiently produced *using x. It is given by the following correspondence:*

$$J\colon \mathbb{R}_+^L \to \mathbb{R}_+^p$$

$$x = (x_1, \dots, x_L) \mapsto J(x) = \left\{q \in \mathbb{R}_+^p : (-x, q) \in E(Y)\right\}$$

Graphs with $p = 2$. Complementary and substitutable outputs.

Properties of production sets

Several assumptions can be made on the production set Y (they may be mutually exclusive). **Graphs.**

Definition (Monotonicity or "free disposal")

With more inputs, one can produce at least as many outputs. Extra amounts of inputs or outputs can be disposed of at no cost:

$$y - \mathbb{R}_+^M \subset Y$$

Definition (Possibility of inaction)

Complete shutdown is possible (no work, no production):

$$0 \in Y$$

Definition (No free lunch)

No output can be produced produced without inputs:

$$Y \cap \mathbb{R}^M_+ \subset \{o\}$$

Definition (Irreversibility)

The technology cannot be reversed. It is impossible to transform an amount of output into the same amount of input that was used to generate it:

$$y \in Y \text{ and } y \neq 0 \;\Rightarrow\; -y \notin Y$$

A specialized firm is a case of irreversibility.

Definition (Additivity or "free entry")

Productive organizations can be replicated:

$$y \in Y \text{ and } y' \in Y \Rightarrow y + y' \in Y$$

Definition (Returns to scale)

- ***Decreasing returns to scale***

$$\forall\ y \in Y \text{ and } \forall\ \lambda \in [0,1],\ \text{we have } \lambda y \in Y$$

- ***Increasing returns to scale***

$$\forall\ y \in Y \text{ and } \forall\ \lambda \geq 1,\ \text{we have } \lambda y \in Y$$

- ***Constants returns to scale***

$$\forall\ y \in Y \text{ and } \forall\ \lambda \geq 0,\ \text{we have } \lambda y \in Y$$

Remark: Observe that a linear technology $+$ fixed costs $\Rightarrow$ IRS.

The case of single-output firm with a homogeneous technology:

Proposition (Returns to scale for a homogeneous technology)

Consider a production function $f : \mathbb{R}^L_+ \to \mathbb{R}_+$ *homogeneous of degree k, i.e.* $\forall \ \lambda \geq 0, \ f(\lambda x) = \lambda^k f(x)$.

We have:

- $k = 1 \ \Leftrightarrow$ *constant returns to scale*
- $k > 1 \ \Leftrightarrow$ *increasing returns to scale*
- $k < 1 \ \Leftrightarrow$ *decreasing returns to scale*

Proof. Example: $f(x_1, x_2) = x_1^\alpha x_2^\beta$ *with* $\alpha > 0$ *and* $\beta > 0$.

Definition (Convexity of the production set)

Y is convex:

$$\forall\ y, y' \in Y \text{ and } \forall\ \alpha \in [0,1],\ \text{we have } \alpha y + (1-\alpha) y' \in Y$$

Remark:

When convexity of Y + possibity of inaction
$\Rightarrow$ **technology with non-increasing returns to scale. Graph.**

Definition (Convexity of the set of required inputs)

$V(q)$ is convex $\forall\ q \in \mathbb{R}_+$:

$$\forall\ x, x' \in V(q) \text{ and } \forall\ \alpha \in [0,1],\ \text{we have } \alpha x + (1-\alpha)x' \in V(q)$$

Remark:

- This is a less stringent assumption (allows for setup costs).
- The convexity of $V(.)$ expresses the reduction in the MRTS along an input isoquant (as substitution between the inputs becomes more difficult). **Graph.**
- Y *convex* $\Rightarrow$ $V(q)$ *convex*.

Proposition

Consider a single-output firm with production function f:

- Y *convex* $\Leftrightarrow$ f *concave.*
- $V(q)$ *convex* $\forall\ q \geq 0$ $\Leftrightarrow$ f *quasiconcave.*

Remark: Since f concave $\Rightarrow$ f quasiconcave, this confirms that Y convex $\Rightarrow$ $V(q)$ convex.

Mathematical complements

Proposition

If a function is homogeneous of degree k, then its derivative is homogeneous of degree $k-1$

- **Proof**.
- **Implication**: For a homogeneous technology, the MRTS does not depend on the scale of production. This remains true for a homothetic technology (i.e. obtained with a strictly increasing transformation of a homogenous function).

Proposition (Euler's theorem)

If f is differentiable and homogeneous of degree k, then

$$\sum_{i=1}^{M} \frac{\partial f(x)}{\partial x_i} \cdot x_i = k \cdot f(x)$$

Proof.

Implications of homogeneity for $Y = F(K, L)$:

- $$k = 1 \;\Rightarrow\; Y = K \cdot \underbrace{\frac{\partial F}{\partial K}}_{marg.\ prod.\ of\ K} + L \cdot \underbrace{\frac{\partial F}{\partial L}}_{marg.\ prod.\ of\ L}$$

- $$k = \underbrace{\varepsilon_K}_{K\ elasticity\ of\ Y} + \underbrace{\varepsilon_L}_{L\ elasticity\ of\ Y}$$

The producer's choice

Two related approaches:

- Cost minimization
- Profit maximization

Cost minimization

Notations

Consider a single-output firm using L inputs:

- $x \in \mathbb{R}^L_+$ is a vector of inputs
- $w \in \mathbb{R}^L_{+*}$ is the vector of exogenous input prices
- $q \in \mathbb{R}$ is the quantity of output to be produced
- $f : \mathbb{R}^L_+ \to \mathbb{R}_+$ is the production function (technology)

- **Problem**:

  Reach a production objective **q** at the minimum cost

- **Program**:

$$\begin{cases} \underset{x \in \mathbb{R}_+^L}{Min} \; w \cdot x \\ s.t. \; f(x) \geq q \end{cases}$$

The conditional factor demand and the cost function

Definition (Conditional factor demand)

The conditional factor demand denoted $\widetilde{x}(w,q)$ is the solution of the cost-minimization program:

$$\left\{ \begin{array}{l} \underset{x \in \mathbb{R}^L_+}{Min} \;\; w \cdot x \\ s.t. \; f(x) \geq q \end{array} \right.$$

Definition (Cost function)

The cost function denoted $C(w,q)$ is the value:

$$C(w,q) = w \cdot \widetilde{x}(w,q)$$

- **Graph**

- In $\widetilde{x}$, the isocost line is tangent to the factor isoquant:

$$|MRTS_{12}| = \frac{\frac{\partial f(\tilde{x})}{\partial x_1}}{\frac{\partial f(\tilde{x})}{\partial x_2}} = \frac{w_1}{w_2}$$

- Some inputs may be fixed in the short term but variable in the long term.

$$C_{LT}(w,q) = C_{ST}(w,q,\widetilde{x}_M(w,q))$$

where $\widetilde{x}_M(w,q)$ is the conditional demand for input M.

Properties of the cost function

Proposition (Properties of the cost function)

A cost function satisfies the following properties:

1. $C(w, q)$ *is homogeneous of degree* 1 *in* w
2. $C(w, q)$ *is increasing in* q
3. $C(w, q)$ *is concave in* w *(**graph and interpretation**)*
4. *If* f *is homogeneous of degree* 1 *(i.e. CRS)* $\Rightarrow$ $C(w, q)$ *is homogeneous of degree* 1 *in* q
5. *If* f *is concave* $\Rightarrow$ $C(w, q)$ *is convex in* q

Relations between technology and production costs

Definition (Average cost)

*The **average cost** is*

$$AC = \frac{C(w, q)}{q}$$

Definition (Marginal cost)

*The **marginal cost** is*

$$Cm = \frac{dC(w, q)}{dq}$$

- Graphs of C, AC and MC in the case of:
  - (i) Decreasing Returns to Scale
    $\Rightarrow$ concave f, convex C, increasing AC and MC
  - (ii) Constant Returns to Scale
    $\Rightarrow$ linear f, linear C, $AC = MC$ constant
  - (iii) Non-convex technology
    $\Rightarrow$ intersection of AC and MC at the minimum of AC
- Observe that:
  - whenever AC increases, we have $MC > AC$
  - whenever AC decreases, we have $MC < AC$

Properties of the conditional factor demand

Proposition (Shepard's Lemma -production version)

The conditional demand for factor k can be obtained by differentiating the cost function:

$$\widetilde{x}_k(w, q) = \frac{\partial C(w, q)}{\partial w_k}$$

Proof.

Proposition (Properties of the conditional factor demand)

$\widetilde{x}(q,w)$ satisfies the following properties:

1. *$\tilde{x}(w,q)$ is homogenous of degree 0 in w*
2. *If f is quasi-concave $\Rightarrow$ $\widetilde{x}(w,q)$ is a convex part of $\mathbb{R}^L_+$*
3. *If f is stricly quasi-concave $\Rightarrow \widetilde{x}(w,q)$ is unique*
4. *If f is homogenous of degree 1 in x*
   $\Rightarrow$ $\widetilde{x}(w,q)$ is homogenous of degree 1 in q
5. *A negative direct price effect:*

$$\forall\ k \in (1,\dots,L),\ \text{we have}: \frac{\partial \widetilde{x}_k(w,q)}{\partial w_k} \leq 0$$

Profit maximization

Notations

Consider a single-output firm using L inputs:

- $x \in \mathbb{R}_{+}^{L}$ is a vector of inputs
- $w \in \mathbb{R}_{+*}^{L}$ is the vector of exogenous input prices
- $f : \mathbb{R}_{+}^{L} \rightarrow \mathbb{R}_{+}$ is the production function (technology)
- $p \in \mathbb{R}_{+*}$ is the price of the output

- **Problem**:

  Maximize the firm's profit given its technology and prices (the firm is "price-taker")

- **Program**:

$$\underset{x \geq 0}{Max}\ p \cdot f(x) - w \cdot x$$

The factor demand, the supply and the profit function

Definition (Factor demand)

The (optimal or derived) factor demand denoted $x(p, w)$ is the solution of the profit-maximization program:

$$\underset{x \geq 0}{Max}\ p \cdot f(x) - w \cdot x$$

Definition (Supply)

The supply function denoted $y(p, w)$ is the production that maximizes profit:

$$y(p, w) = f(x(p, w))$$

Definition (Profit function)

The profit function denoted $\Pi(p, w)$ is the value of the profit-maximization program:

$$\Pi(p, w) = \underset{x \geq 0}{Max}\; p \cdot f(x) - w \cdot x = p \cdot f(x(p, w)) - w \cdot x(p, w)$$

It yields the maximal profit for any price vector (p, w)

Remarks:

- If x^* (input vector) is a solution of the profit-maximization program, we have for each factor k:

$$\underbrace{p \cdot \frac{\partial f}{\partial x_k}(x^*)}_{\text{marginal productivity (value)}} = \underbrace{w_k}_{\text{price of input k}}$$

- The program can **equivalently** be written:

$$\underset{q}{Max}\ p \cdot q - C(w, q)$$

If q^* (output) is a solution of the profit-maximization program re-stated above, then it satisfies:

$$\underbrace{p}_{\text{output price (marginal sale)}} = \underbrace{\frac{\partial C}{\partial q}(q^*, w)}_{\text{marginal cost}}$$

- Hence the **profit-maximizing rule**:

  Produce in the region where the marginal cost is increasing (with $MC > AC$) until it equals the product's price ($MC = p$).

- If $MC < p$ then profit can be increased by producing more.

- If $MC > p$ then profit can be increased by producing less.

- The **supply curve** is (partially) identified to the **marginal cost curve.**

Graphic illustration: the supply curve

Consider the profit-maximization program (second version):

$$\Pi(q) = \underset{q}{Max}\; p \cdot q - C(w, q)$$

There are three necessary conditions for an optimal solution q^* to exist:

$$\begin{cases} \Pi(q^*) \geq 0 \\ \Pi'(q^*) = 0 \\ \Pi''(q^*) \leq 0 \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} p \geq \frac{C(q^*,w)}{q^*} = AC(q^*) \\ p = \frac{\partial C(q^*,w)}{\partial q} = MC(q^*) \\ \frac{\partial^2 C}{\partial q^2}(q^*, w) \geq 0 \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} \textit{price} > \textit{average cost} \\ \textit{price} = \textit{marginal cost} \\ \textit{locally convex cost, i.e. increasing marginal cost} \end{cases}$$

The marginal cost curve determines the supply curve.

Graphic illustration: the supply curve (continued)

Examples

- Strictly convex technology
- Non-convex technology
- Graphic representation of profit

The profit-maximization program in the general case

Consider a non-specialized firm and a production plan $y \in Y$.

Recall that the price vector (for both inputs and outputs) is denoted p.

The firm's program can be written:

$$\underset{y \in Y}{Max}\ p \cdot y$$

Definition (Supply correspondence)

The supply correspondence denoted $y(p)$ is the program's solution. It gives the quantities of supplied goods (outputs) and demanded factors (inputs) that maximize profit.

Definition (Profit function)

The profit function denoted $\Pi(p)$ is the value of the program:

$$\Pi(p) = \underset{y \in Y}{Max}\ p \cdot y$$

Remark:

The supply correspondence is $y(p) = \{y \in Y : \ p \cdot y = \Pi(p)\}$

Properties of the profit function

Proposition (Properties of the profit function)

The profit function satisfies the following properties:

- $\Pi(p)$ *is homogenous of degree* 1
- $\Pi(p)$ *is convex (graph and interpretation)*

Proposition (Hotelling's Lemma)

The optimal demand or supply for any good k can be derived from the profit function:

$$y_k(p) = \frac{\partial \Pi(p)}{\partial p_k}$$

Proof.

Properties of the supply correspondence

Proposition

$y(p)$ satisfies the following properties:

1. *$y(p)$ is homogeneous of degree 0.*
2. *If Y is convex $\Rightarrow$ $y(p)$ is convex.*
   If Y is strictly convex $\Rightarrow$ $y(p)$ is unique.
3. *Positive direct price effect $\frac{\partial y_k(p)}{\partial p_k} \geq 0$ (Law of supply)*
   If k is an output, supply increases in the price.
   If k is an input, demand decreases in the price.

Summary

Consider a single-output firm using L inputs:

- $x \in \mathbb{R}_+^L$ is a vector of inputs
- $w \in \mathbb{R}_{+*}^L$ is the vector of exogenous input prices
- $f : \mathbb{R}_+^L \to \mathbb{R}_+$ is the production function (technology)
- $p \in \mathbb{R}_{+*}$ is the price of the output

The profit-maximization program

$$\underset{x_i \geq 0}{Max}\ p \cdot f(x_1, \ldots, x_L) - \sum_{i=1}^{L} w_i x_i$$

- optimal input demands: $x_i^*(p, w)$
- optimal output supply: $y^*(p, w) = f(x_1^*, \ldots, x_L^*)$
- profit function: $\Pi(p, w) = p \cdot y^* - \sum_{i=1}^{L} w_i x_i^*$
- Hotelling's lemma (with the notations of the single-product case):

$$y^*(p, w) = \frac{\partial \Pi(p, w)}{\partial p}$$

$$x_i^*(p, w) = -\frac{\partial \Pi(p, w)}{\partial w_i}$$

The cost-minimization program

$$\begin{cases} \underset{x_i \geq 0}{Min} \sum_{i=1}^{L} w_i x_i \\ s.t.\ f(x_1, \ldots, x_L) \geq q \end{cases}$$

- conditional factor demands: $\widetilde{x}_i(q, w)$
- cost function: $C(q, w) = \sum_{i=1}^{L} w_i \widetilde{x}_i$
- Shephard's lemma (with the notations of the single-product case):

$$\widetilde{x}_i(q, w) = \frac{\partial C(q, w)}{\partial w_i}$$

Some duality relations

- Conditional factor demand and optimal demand:

$$x^*(p, w) = \widetilde{x}(w, y^*(p, w))$$

- The geometry of duality: the slope of an isoquant is the ratio of factor prices while the slope of an isocost curve is the ratio of factor quantities. **Proof.**

Addendum: aggregating firms' supplies

Proposition (Aggregating supplies)

Let p be the price of a good potentially produced by N firms.
If $y_i^s(p)$ is firm i's supply (as a function of p),
then $\sum_{i=1}^{M} y_i^s(p)$ is the aggregate supply.