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# Chapter 5 - Pareto optimality in exchange and production economies

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# Modeling an exchange economy

**Assumptions and notations:**

- $I = \{1, .., i, ..., N\}$ is a set of $N$ individuals.
- Each individual faces a consumption set $X_i$ and has a utility function $u_i : X_i \to \mathbb{R}$.
- $G = \{1, ..., g, ..., M\}$ is a set of $M$ perfectly divisible goods. Hence, $X_i = \mathbb{R}_+^M$.
- $W = (W_1, ..., W_g, ..., W_M)$ is the vector of total initial endowments of goods in the economy, where $W_g$ is the total initial endowment of good $g$ (a non-negative value).

### Definition (Exchange economy)

*An **exchange economy** is a 4-uple:*

$$\mathcal{E} = \left(I,\ (X_i)_{i\in I}\ , (u_i)_{i\in I},\ W\right)$$

### Definition (Allocation)

*An **allocation** is the vector that gives the distribution of all goods among all individuals :*

$$x = (x^1, x^2, ..., x^N) \in \prod_{i=1}^{N} X_i$$

*where $x^i = (x^i_1, x^i_2, ..., x^i_M)$ is individual $i$'s consumption of all goods, with $x^i_g$ the quantity of good $g$ allocated to individual $i$.*

In extended form, we can write:

$$x = (\underbrace{x_1^1, x_2^1, ..., x_M^1}_{x^1}, ..., \underbrace{x_1^i, x_2^i, ..., x_M^i}_{x^i}, ..., \underbrace{x_1^N, x_2^N, ..., x_M^N}_{x^N})$$

## Definition (Feasible allocation)

*An allocation is* ***feasible*** *if:*

$$\sum_{i \in I} x^i \leq W$$

*which also writes:*

$$\sum_{i \in I} x_g^i \leq W_g \quad \forall g \in G$$

**Graphic representation in an Edgeworth box.**
$(M = 2, N = 2)$

# Definition of Pareto optima

## Definition (Pareto optimum)

*An allocation* $x^* = ({x^1}^*, ..., {x^i}^*, ..., {x^N}^*)$ *is a **Pareto optimum** (denoted **PO**) if:*

- *It is feasible:*

$$\sum_i {x_g^i}^* \leq W_g \quad \forall g \in M$$

- $\nexists$ *x feasible such that:*

$$\begin{cases} u_i(x^i) \geq u_i({x^i}^*) & \forall i \in I \\ \exists j \in I, & u_j(x^j) > u_j({x^j}^*) \end{cases}$$

**Interpretation**:

- Starting from $x^*$, another feasible allocation that would increase the well-being of one agent without decreasing the well-being of another one does not exist.
- Pareto optimality is only an **efficiency criterion** (sometimes labeled "Pareto efficiency"). It is **not necessariy equitable**: an allocation that grants all resources to only one agent is always a Pareto Optimum. Efficiency is not necessarily desirable when it occurs at the expense of equity.

**Interpretation** (continued):

- Graph of a **Pareto frontier**.
- Pareto optimality is a limited concept. It defines as efficient **social states that are not-unanimously rejected.** Indeed, there will always exist at least one agent in a Pareto optimum that will prefer this optimum to any other alternative allocation (proof: by contradiction).
- **PO are just allocations.**

# Characterization of Pareto optima

**Assumption**: all individual utility functions are quasi-concave and twice continuously differentiable.

## Proposition (Characterization of PO)

- *Any allocation $x^*$ solution of the following program:*

$$\begin{cases} Max\ u_1(x_1^1, x_2^1, ..., x_M^1) \\ s.t. \quad x = (x^1, x^2, ..., x^N) \text{ is feasible} \\ \qquad u_i(x_1^i, x_2^i, ..., x_M^i) \geq \bar{u}_i \quad i = 2, ..., N \end{cases}$$

  *is a Pareto optimum.*
- *Reciprocally, any PO is a solution of the above program (written with the adequate vector of utility-level parameters $(\bar{u}_2, ..., \bar{u}_N)$).*

**Characterization of interior solutions** Here, we will focus on interior solutions only. The Lagrangian writes:

$$\begin{aligned} \mathcal{L} &= u_1(x_1^1, ..., x_g^1, ..., x_M^1) \\ & -\sum_{i=2}^{N} \lambda_i \left[ \bar{u}_i - u_i(x_1^i, x_2^i, ..., x_M^i) \right] \\ & -\sum_{g=1}^{M} \mu_g \left[ \sum_{i=1}^{M} x_g^i - W_g \right] \end{aligned}$$

The F.O.C. are:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_g^i} = 0 \quad \forall g \in G, \ \forall i \in I \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \quad \forall i \in I \\ \frac{\partial \mathcal{L}}{\partial \mu_g} = 0 \quad \forall g \in G \end{cases}$$

Simple calculations yield the following characterization:

$$\underbrace{\frac{\partial u_i / \partial x_g^i}{\partial u_i / \partial x_{g'}^i}}_{MRS_{gg'}^{\ i}} = \underbrace{\frac{\partial u_{i'} / \partial x_g^{i'}}{\partial u_{i'} / \partial x_{g'}^{i'}}}_{MRS_{gg'}^{\ i'}} \quad \forall i, i' \in I, \ \forall g, g' \in G$$

**Proposition (Interior solutions)**

*The interior PO of an exchange economy are characterized by:*

$$MRS^{i}_{gg'} = MRS^{i'}_{gg'} \quad \forall i, i' \in I, \ \forall g, g' \in G$$

**Interpretation**:

- In Pareto-optimal allocations, all individuals have the same MRS for any given couple of goods.
- If it were not the case, then a reallocation of the goods between two agents could increase the utility of one agent without decreasing the utility of the other (there would exist a mutually advantageous exchange, contradiction with OP).

## Graphic representation in an Edgeworth box

### Definition (Pareto-optimal allocation)

*An allocation $x^* = (x^{1*}, x^{2*})$ in the Edgeworth box is Pareto optimal if there is no other allocation $x = (x^1, x^2)$ in the Edgeworth box with $x^i \underset{i}{\succsim} x^{i*}$ for $i = 1, 2$ and $x^i \underset{i}{\succ} x^{i*}$ for some $i$ (i.e. either for $i = 1$ or for $i = 2$).*

### Definition (Pareto set)

*The set of all Pareto optimal allocations is called the* ***Pareto set****. It is the set of* ***all tangency points between the two consumers' indifference curves*** *(where MRS are equalized).*

**Definition (Contract curve)**

*The part of the Pareto set* ***where both consumers do at least as well as their initial endowments*** *is called the* ***contract curve****.*

**Graph.**

# Modeling an exchange and production economy

**Assumptions and notations**:

- $I = \{1, .., i, ..., N\}$ is a set of $N$ consumers.
- Each individual faces a consumption set $X_i$ and has a utility function $u_i : X_i \to \mathbb{R}$.
- $G = \{1, ..., g, ..., M\}$ is a set of $M$ perfectly divisible goods. Hence, $X_i = \mathbb{R}_+^M$.
- $W = (W_1, ..., W_g, ..., W_M)$ is the initial endowment of goods in the economy (vector), where $W_g$ is the initial endowment of good $g$ (a non-negative value).

**Assumptions and notations** (continued):

- $J = \{1, .., j, ..., Q\}$ is a set of $Q$ non-specialized firms
- $Y_j = \{y \in \mathbb{R}^M : F_j(y) \leq 0\}$ is the production set of firm $j$, with $F_j$ firm $j$'s transformation function.
- $y_g^j$ is the quantity of good $g$ produced by firm $j$ (if $y_g^j > 0$) or utilizied by firm $j$ (if $y_g^j < 0$).

**Definition (Exchange and production economy)**

*An* ***exchange and production economy*** *is a 6-uple:*

$$E = (I,\ (X_i)_{i \in I},\ (u_i)_{i \in I},\ J,\ Y_j,\ W)$$

## Definition (Allocation)

*An **allocation** is the vector that gives the distribution of all consumed and produced goods among all individuals and firms:*

$$(x, y) = (x^1, x^2, ..., x^N, y^1, y^2, ..., y^Q) \in \prod_{i=1}^{N} X_i \times \prod_{j=1}^{Q} Y_j$$

*where :*

- $x^i = (x^i_1, x^i_2, ..., x^i_M)$ *is individual* $i$*'s consumption of all goods, with* $x^i_g$ *the quantity of good* $g$ *allocated to individual* $i$ *(with* $x^i_g \geq 0$*).*
- $y^j = (y^j_1, y^j_2, ..., y^j_M)$ *is firm* $j$*'s production and utilization of all goods, with* $y^j_g$ *is the quantity of good* $g$ *produced by firm* $j$ *(if* $y^j_g > 0$*) or used by firm* $j$ *(if* $y^j_g < 0$*).*

**Remark**:

- In extended form, we can write:

$$(x, y) = (\ldots, \underbrace{x_1^i, x_2^i, \ldots, x_M^i}_{x^i}, \ldots, \underbrace{y_1^j, y_2^j, \ldots, y_M^j}_{y^j}, \ldots)$$

- Observe that with our convention, consumptions are $> 0$ for consumers but $< 0$ for firms.

## Definition (Feasible allocation)

*An allocation is **feasible** if:*

- *Utilizations (consumptions) are no greater than resources (productions + endowments):*

$$\underbrace{\sum_{i=1}^{N} x_g^i}_{consumption} \leq \underbrace{W_g}_{endowment} + \underbrace{\sum_{j=1}^{Q} y_g^j}_{production} \quad \text{for } g = 1, ..., M$$

- *Production is feasible given the technology:*

$$F_j(y_1^j, y_2^j, ..., y_M^j) \leq 0 \quad \text{for } j = 1, ..., Q$$

**Example**:

- $N = 2$, two consumers: 1 and 2
- $M = 2$, two goods: 1 and 2, with initial endowments $W_1$ and $W_2$
- $Q = 2$, two single-product firms: 1 (producing good 1) and 2 (producing good 2) using capital and labor, with initial endowments $K$ and $L$

**Example** (continued):

- In this example, an allocation

$$z = (\underbrace{x_1^1, x_2^1}_{x^1}, \underbrace{x_1^2, x_2^2}_{x^2}, y_1, y_2, L_1, L_2, K_1, K_2)$$

is feasible if:

$$\begin{cases} x_1^1 + x_1^2 & \leq W_1 + y_1 \\ x_2^1 + x_2^2 & \leq W_2 + y_2 \\ L_1 + L_2 & \leq L \\ K_1 + K_2 & \leq K \\ y_1 & \leq f_1(K_1, L_1) \\ y_2 & \leq f_2(K_2, L_2) \end{cases}$$

## Characterization of Pareto optima

**Proposition**

- *Any allocation* $(x^*, y^*)$ *solution of the following program:*

$$\begin{cases} Max\ u_1(x_1^1, x_2^1, ..., x_M^1) \\ s.t. \quad (x, y) = (x^1, x^2, ..., x^N, y^1, y^2, ..., y^Q) \text{ is feasible} \\ \qquad u_i(x_1^i, x_2^i, ..., x_M^i) \geq \bar{u}_i \quad i = 2, ..., N \end{cases}$$

  *is a Pareto optimum.*
- *Reciprocally, any PO is a solution of the above program (written with the adequate vector of utility-level parameters* $(\bar{u}_2, ..., \bar{u}_N)$*).*

## Characterization of interior solutions

Here again, we will focus on interior solutions only. The

Lagrangian writes:

$$\begin{aligned}\mathcal{L} &= u_1(x_1^1, ..., x_g^1, ..., x_M^1) \\ &- \sum_{i=2}^{N} \lambda_i \left[ \bar{u}_i - u_i(x_1^i, x_2^i, ..., x_M^i) \right] \\ &- \sum_{g=1}^{M} \mu_g \left[ \sum_{i=1}^{M} x_g^i - W_g - \sum_{j=1}^{Q} y_g^j \right] \\ &- \sum_{j=1}^{Q} \nu_j \left[ F_j(y_1^j, y_2^j, ..., y_M^j) - 0 \right]\end{aligned}$$

The F.O.C. are:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_g^i} = 0 \quad \forall g \in G, \ \forall i \in I \\ \frac{\partial \mathcal{L}}{\partial y_g^j} = 0 \quad \forall g \in G, \ \forall j \in J \\ \frac{\partial \mathcal{L}}{\partial \lambda^i} = 0 \quad \forall i \in I \\ \frac{\partial \mathcal{L}}{\partial \mu_g} = 0 \quad \forall g \in G \\ \frac{\partial \mathcal{L}}{\partial \nu_j} = 0 \quad \forall j \in J \end{cases}$$

## Proposition

*The interior PO of an exchange and production economy are characterized by:*

- *Equalization of MRS (consumption allocation efficiency)*

$$MRS^{i}_{g,g'} = MRS^{i'}_{g,g'} \quad \forall i, i' \in I, \quad \forall g, g' \in G$$

- *Equalization of MRT (production allocation efficiency)*

$$MRT^{j}_{g,g'} = MRT^{j'}_{g,g'} \quad \forall j, j' \in J, \quad \forall g, g' \in G$$

- *Equalization of MRS and MRT (link between consumption and production)*

$$MRS^{i}_{g,g'} = MRT^{j}_{g,g'} \quad \forall i \in I, \forall j \in J, \quad \forall g, g' \in G$$

**Coming back to the $M = N = Q = 2$ example**:
An allocation $z = (x_1^1, x_2^1, x_1^2, x_2^2, y_1, y_2, L_1, L_2, K_1, K_2)$ is a PO it it solves:

$$\begin{cases} \quad \text{Max } u_1(x_1^1, x_2^1) \\ u_2(x_1^2, x_2^2) \geq \overline{u}_2 \\ x_1^1 + x_1^2 \leq W_1 + y_1 \\ x_2^1 + x_2^2 \leq W_2 + y_2 \\ L_1 + L_2 \leq L \\ K_1 + K_2 \leq K \\ y_1 = f_1(K_1, L_1) \\ y_2 = f_2(K_2, L_2) \end{cases}$$

**Calculations. Graphs.**